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COMMENT

Periodic solutions of the DABO equation as a sum of repeated solitons

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Abstract. It is shown that the periodic solution of the non-linear DABO equation can be written as an infinite sum of an equally spaced row of indentical Lorentzian solitons. This may be considered as a generalisation of a similar result which has been found for the KdV equation and is related to the clean-interaction properties of colliding solitons under non-linear coupling.

A remarkable property of the Korteweg-de Vries (KdV) equation is that its periodic solutions (cnoidal waves) may be represented as exact sums of equally spaced identical solitons. Such a representation was first obtained by Toda (1975) as a by-product of a more general discussion of the 'Toda lattice' by using an infinite-product method. The importance of this result and its relevance to the 'clean-interaction' and the 'non-destructive' properties of colliding solitons of the KdV equation under non-linear coupling, was recently rediscovered by Boyd (1984) and Whitham (1984). Boyd used the Poisson summation formula and Whitham employed the method of partial fractions to prove that cnoidal waves may indeed be represented as an infinite sum of the spatially repeated 'sech^2' type solitary wave solutions of the KdV equation. The purpose of this comment is to show that the same property also holds for the Davis-Acrivos-Benjamin-Ono (DABO) equation. Some general properties of this equation, obtained by using an inverse scattering transform scheme, are discussed in a paper by Santini *et al* (1984). The same paper also includes a comprehensive list of references on this important integrodifferential equation.

The one-dimensional DABO equation

$$\rho_t + \alpha\rho\rho_x - \beta H(\rho_{xx}) = 0 \qquad (1)$$

where H denotes the Hilbert transform defined by

$$H\{f(x, t)\} = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(\xi, t)}{x - \xi} \, d\xi \qquad (2)$$

and α, β are positive constants, was first given by Davis and Acrivos (1967) and by Benjamin (1967); both of these investigations concerned two-dimensional internal waves propagating under the free surface of an infinitely deep fluid. This non-linear integrodifferential equation (1) was later investigated by Ono (1975) in the same context, who derived four conservation laws for (1) and applied them to estimate soliton evolutions from a given initial disturbance. It was shown analytically by

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Benjamin, and numerically by Davis and Acrivos, that the DABO equation has a simple steady solitary wave solution in the form of a Lorentzian (algebraic) shape, unlike the hyperbolic secant function which corresponds to the KdV equation. The elevation of the solitary wave in a reference frame moving with the wave, $\xi = x - Vt$, $(V > 0)$ is given by

$$\rho_s(\xi) = -a\lambda^2/(\xi^2 + \lambda^2) \tag{3}$$

where

$$a = 4V/\alpha \qquad \lambda = \beta/V. \tag{4}$$

In addition to the solitary wave, (3), the DABO equation also yields a periodic wavetrain solution with period $2L$ (Benjamin 1967, Ono 1975):

$$\rho_p(\xi) = \frac{-a\delta^2/2}{1 - \mu\cos(\pi\xi/L)} \tag{5}$$

which is a two-parameter solution of (1) with

$$\delta = \pi\beta/VL \qquad \mu^2 + \delta^2 = 1. \tag{6}$$

For $L \to \infty$ (infinite period) the periodic solution (5) reduces to the solitary wave (3) and the case $L \to 4\pi\beta/a\alpha$ corresponds to infinitesimal waves.

It is rather surprising to note that the periodic wave (5) may also be expressed in terms of an infinite sum of equally spaced indentical solitons (3), i.e.

$$\frac{\delta^2}{1 - \sqrt{1-\delta^2}\cos(\pi\xi/L)} = k \sum_{n=-\infty}^{\infty} \frac{\gamma^2}{(\xi - 2nL)^2 + \gamma^2} \tag{7}$$

where

$$\gamma = (L/\pi)\tanh^{-1}\delta \qquad k = 2\delta/\tanh^{-1}\delta. \tag{8}$$

For infinitely long period $L \to \infty$, equations (4), (6) and (8) yield $\gamma \to \lambda$, $k \to 2$ and (7) reduces to $\rho_p(\xi) \to \rho_s(\xi)$.

The above identity (7) may be readily verified by considering the following contour integral in the complex plane:

$$I = \oint_C \frac{1}{(\xi - 2ZL)^2 + \gamma^2} \frac{dZ}{\exp(2\pi i Z) - 1} \tag{9}$$

where $Z \triangleq \xi + i\gamma$ and C denotes a circle of radius R such that $R = |Z|$. For $R \to \infty$ we have $I = O(1/R^2) \to 0$ and the residue theorem implies that the sum of all residues of the integrand (9) lying within C is zero. Thus

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\xi - 2nL)^2 + \gamma^2} + 2\pi i \operatorname{res}\left(\frac{\xi + i\gamma}{2L}\right) + 2\pi i \operatorname{res}\left(\frac{\xi - i\gamma}{2L}\right) = 0 \tag{10}$$

and a straightforward computation of the contributions from the two residues in (10) finally results in (7) and (8).

The same result (7) of course applies to the Lorentzian solitary wave solution (in addition to the 'sech' solution) of the modified KdV equation found by Zabusky (1967), where it is still astonishing to find that solitary wave solutions of the non-linear DABO equation may be simply added together to give the periodic solution of the same equation.

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